Neuronal and Network Dynamics

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About me

i. Call me “Indra”: one of your tutors at CCNS 2026
ii. Postdoctoral fellow in computational neuroscience @ University College Dublin

iii. B.Sc. Physics
iv. M.Sc. Physics
v. Ph.D. Applied Mathematics

vi. Visiting fellow in neuroscience @ Trinity College Dublin
vii. Write softwares in 🐍 and R
viii. Research in dynamical systems, network science, and computational science
ix. Find me at https://indrag49.github.io/

Overview

- Modelling neuron dynamics
- Firing rate models
- Fixed points and stability
- Single post-synaptic neural network
- Two-population firing rate model
- FP and stability of WC model
- Phase plane analysis
- Spiking neural model
- Spiking neural network
- Mean field model

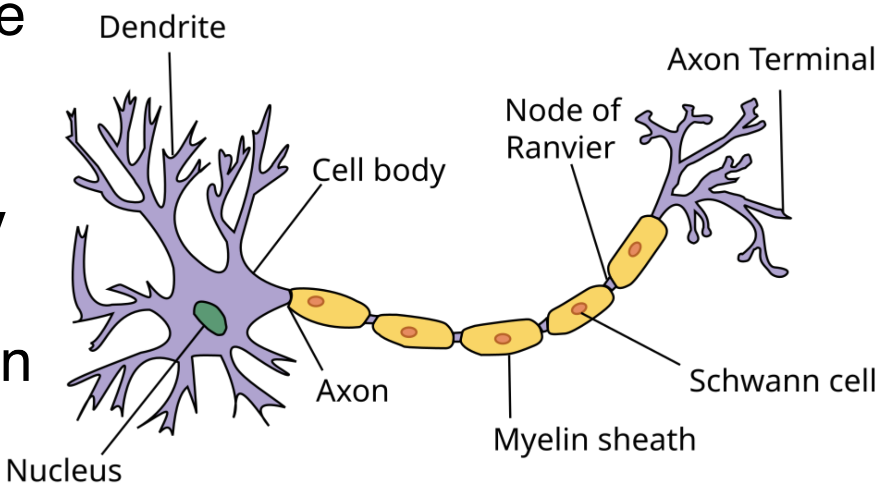
Modelling neuron dynamics

1. Neurons evolve over time.

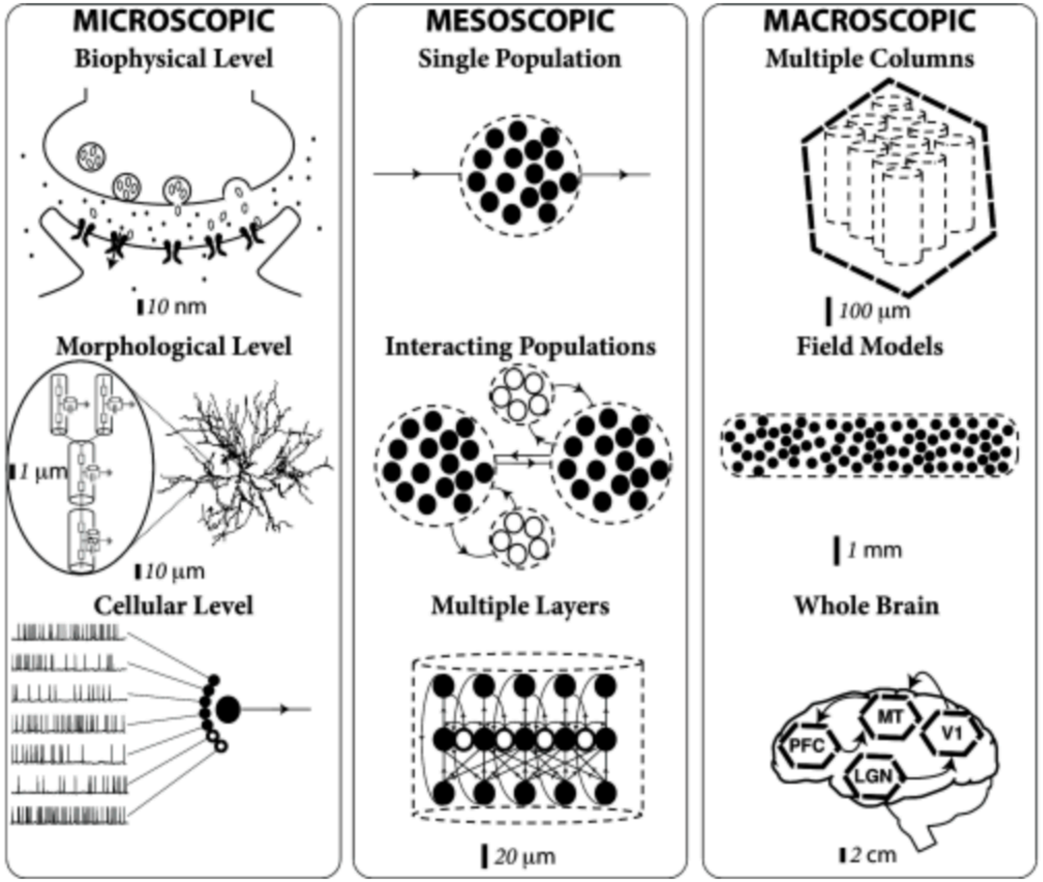
2. Examples of dynamical variables are
   (a) membrane voltage
   (b) firing rate
   (c) synaptic current
   (d) population activity

3. Mathematically written as an ODE:

$$\frac{dx}{dt} = f(x, t)$$



Source: https://commons.wikimedia.org/wiki/File%3ANeuron.svg



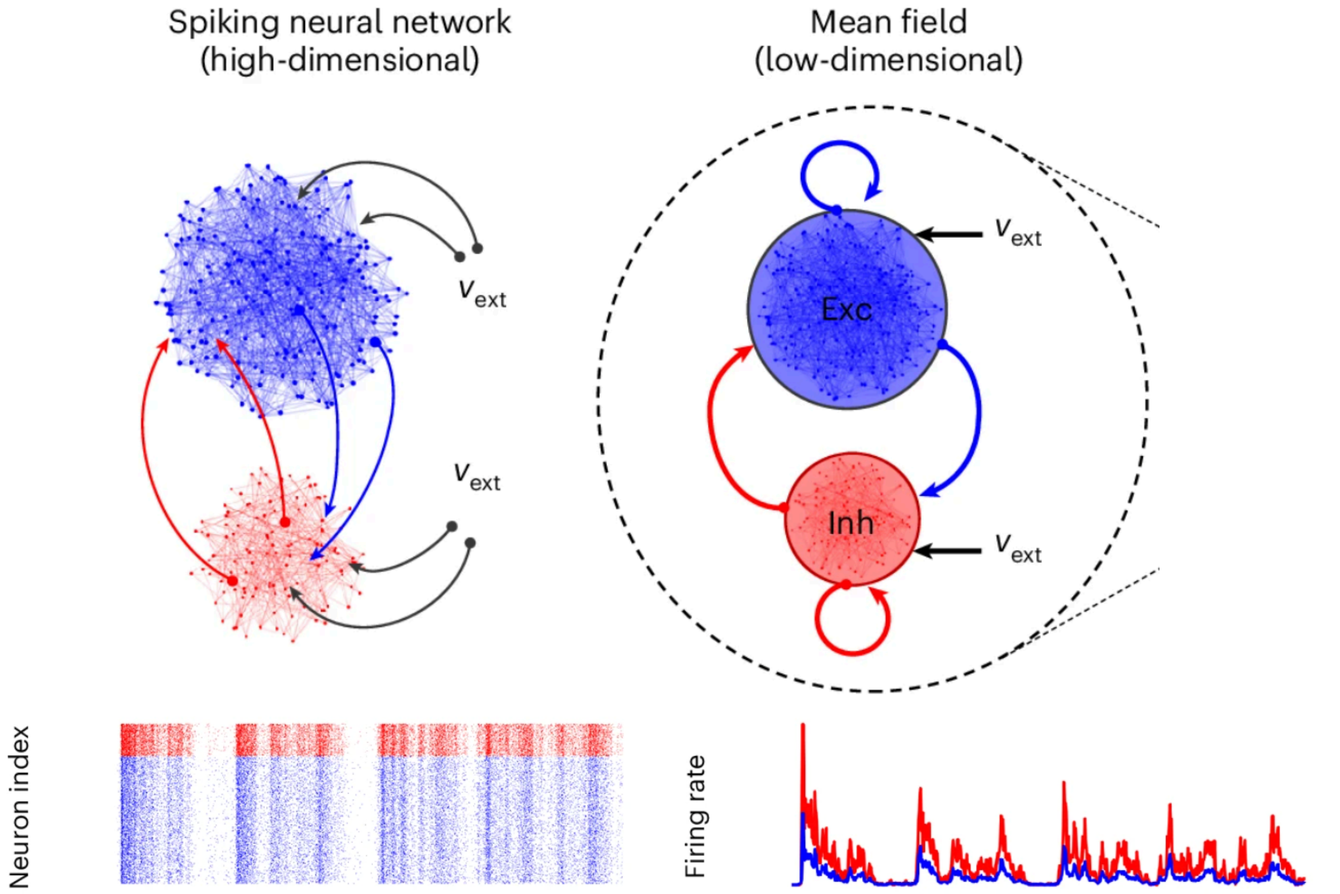
Source: https://neuronaldynamics.epfl.ch/online/Ch12.html

Firing rate models

1. Rate models describe average activity.
2. Not individual spikes.
3. A simple model is

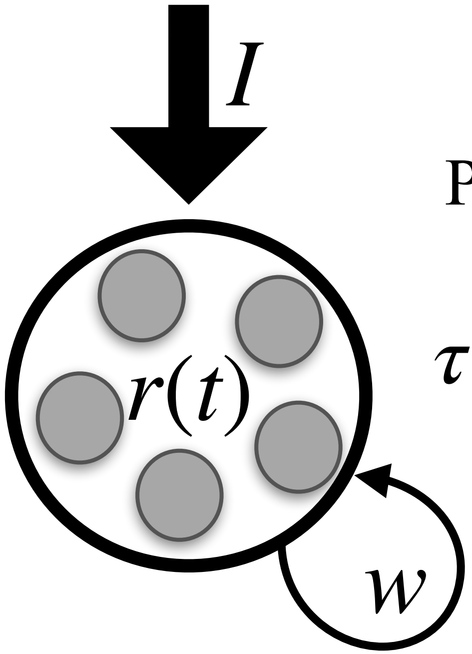
$$\boxed{\tau \frac{dr}{dt} = -r + \sigma(wr + I)}$$

where
$r(t)$: firing rate
τ : time constant
w : recurrent coupling strength
$I(t)$: external input current
$\sigma()$: nonlinear activation function



Source: “A computational approach to evaluate how molecular mechanisms impact large-scale brain activity." *Nature Computational Science* 5.5 (2025): 405-417.

Firing rate models



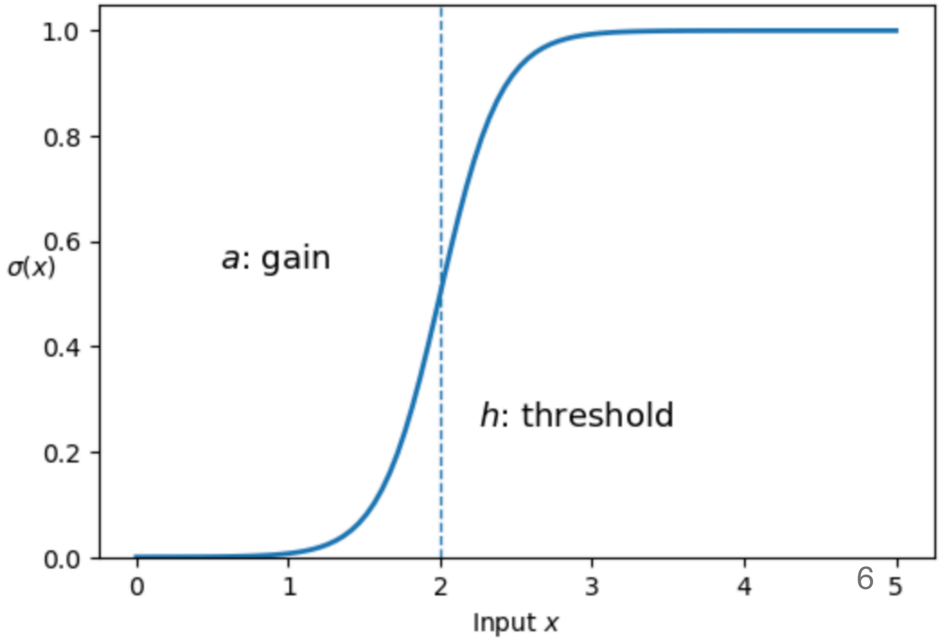
Population firing rate:

$$\tau \frac{dr}{dt} = -r + \sigma(wr + I)$$

Sigmoidal firing rate function:

$$\sigma(x) = \frac{1}{1 + e^{-a(x-h)}}$$

a : gain, h : threshold



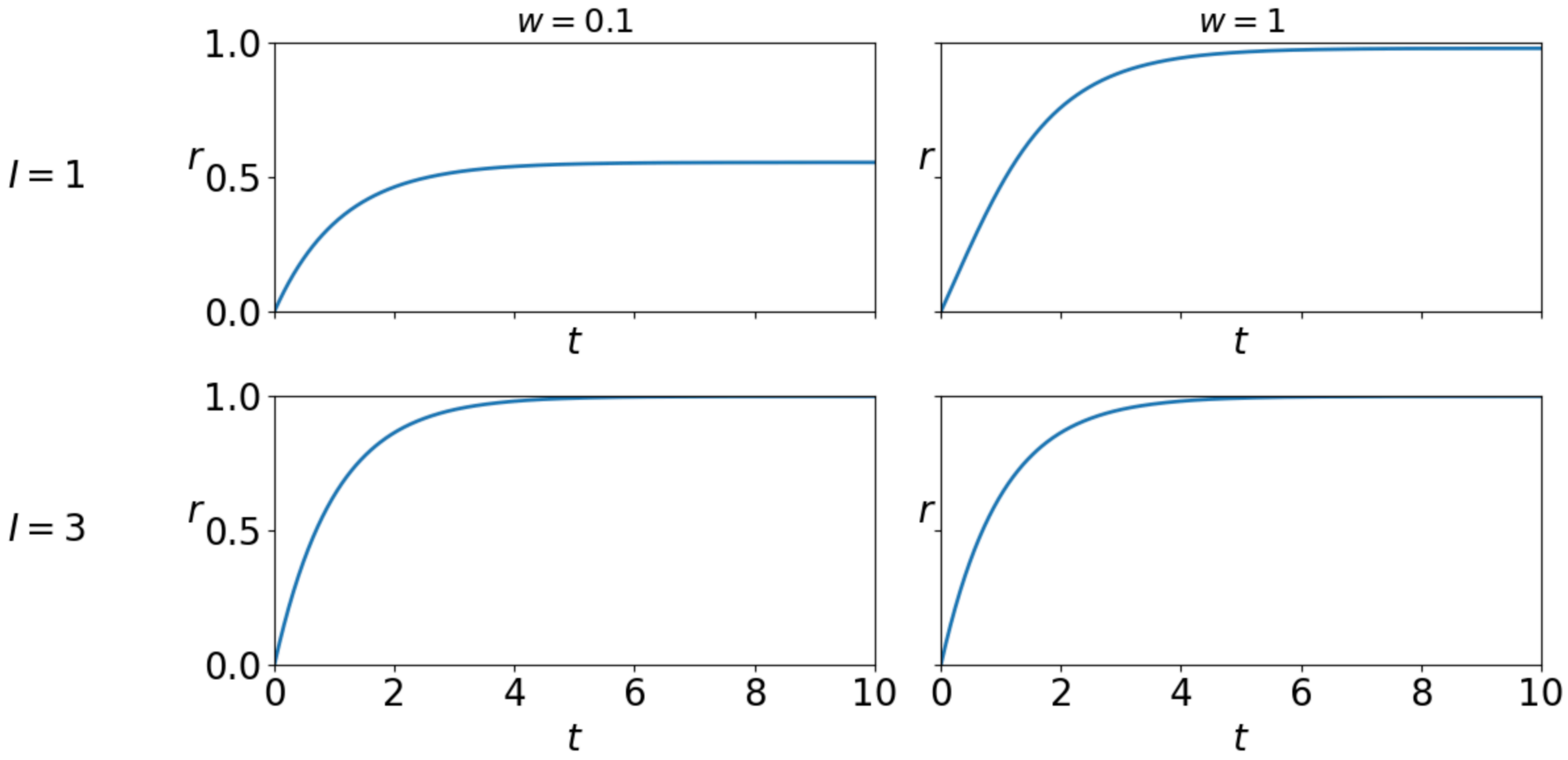
Firing rate models

1. The rate r approaches its steady state/ fixed point (r^*) with time

2. The fixed point is dependent on the system parameters like I, w, etc.

3. We can compute r^* mathematically.

4. For the plots lets look at jupyter Notebook



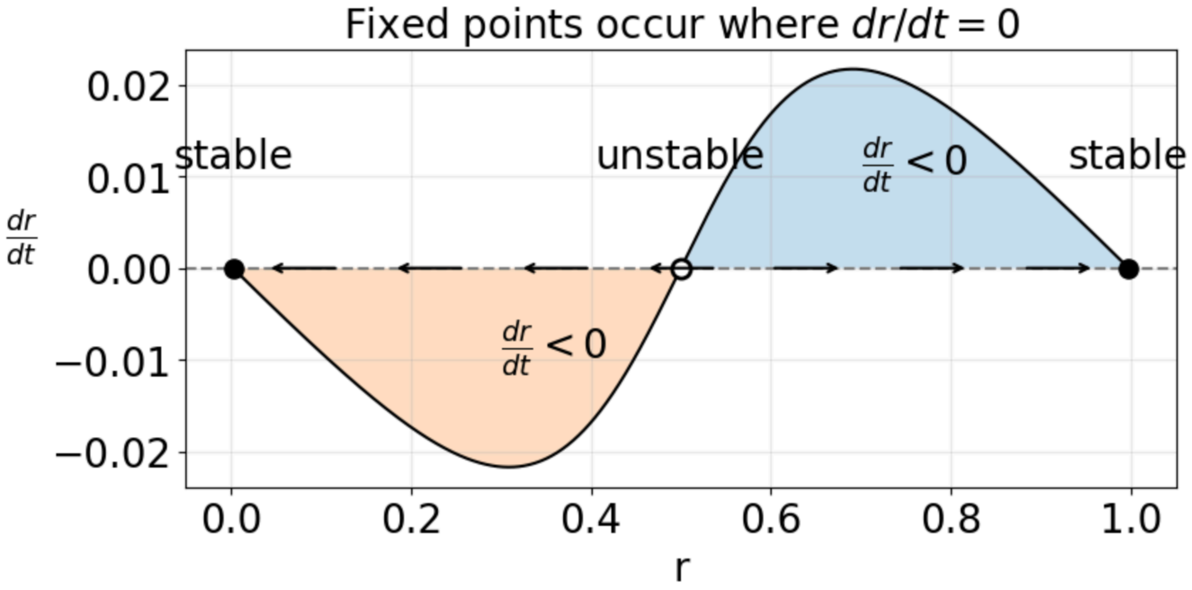
Fixed points and stability of rate models

1. A fixed point occurs when $\frac{dr}{dt} = 0$

2. For the rate model: $-r + \sigma(wr + I) = 0$
satisfying $r^* = \sigma(wr^* + I)$

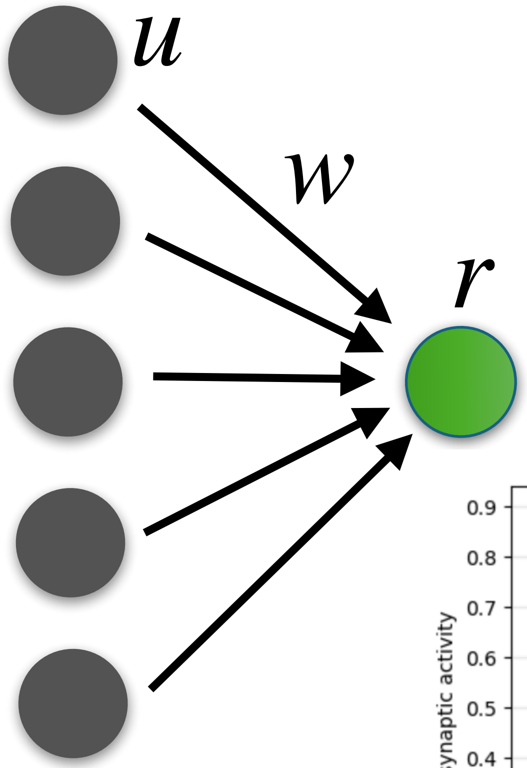
3. Compute the derivative to check for stability

$$\frac{d}{dr}\left[\frac{-r + \sigma(wr + I)}{\tau}\right] = \frac{-1 + w\frac{d}{dr}(\sigma(wr + I))}{\tau}$$



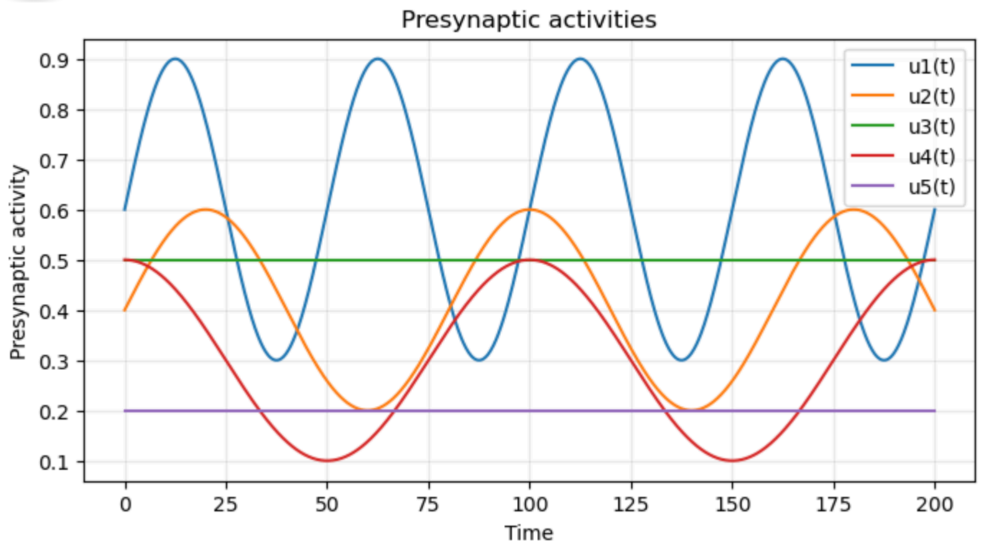
4. If $\frac{dr}{dt} > 0$: flow is to the right, if $\frac{dr}{dt} < 0$: flow is to the left. Fixed point is stable if $\frac{df}{dr}\Big|_{r=r^*} < 0$, else unstable. See jupyter Notebook

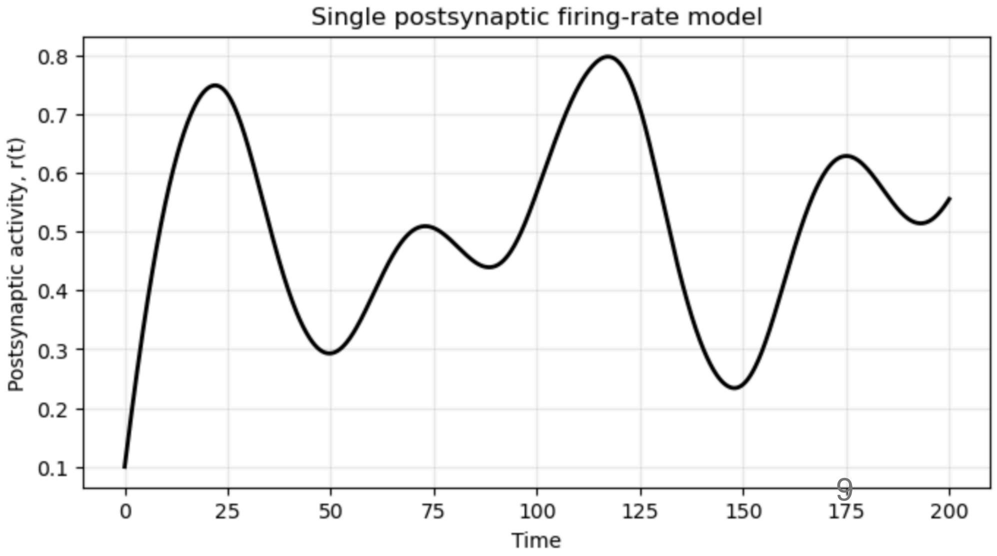
Single post-synaptic neural network



$$\tau \frac{d\mathbf{r}}{dt} = -\mathbf{r} + \sigma(\mathbf{w} \cdot \mathbf{u})$$

1. u : activity of presynaptic neurons
2. r : activity of postsynaptic neurons
3. w : strength of weights between the presynaptic and postsynaptic weights





Two population firing rate model

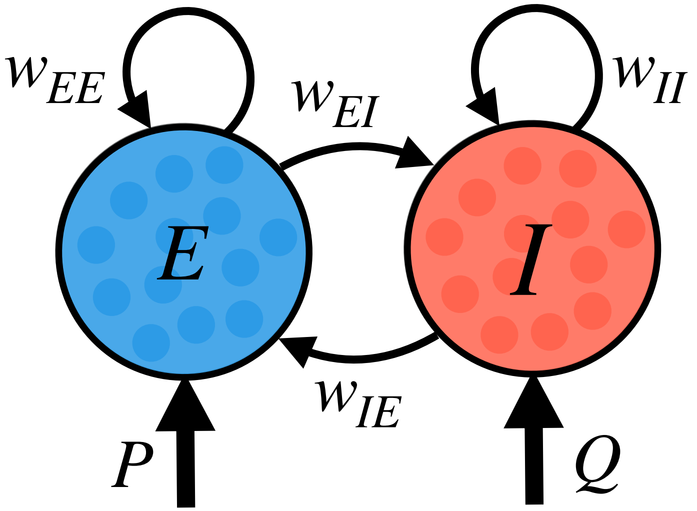
1. Population of excitatory (E) and inhibitory (I) neurons coupled to each other

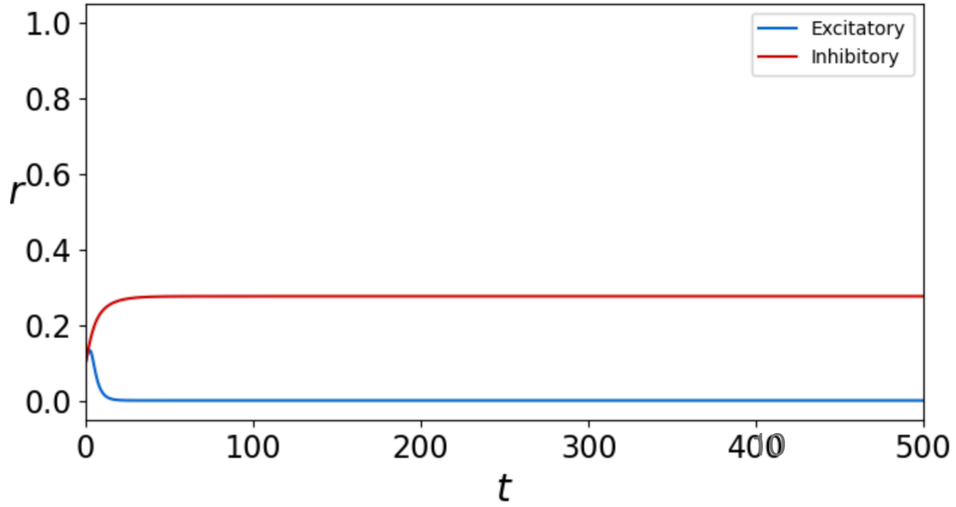
2. Called Wilson-Cowan rate model:

$$\tau_E \frac{dE}{dt} = -E + \sigma(w_{EE}E - w_{EI}I + P),$$

$$\tau_I \frac{dI}{dt} = -I + \sigma(w_{IE}E - w_{II}I + Q),$$

here $E(t), I(t)$: activities of the excitatory and inhibitory populations, τ_E, τ_I are the time constants, w is the connection weight, with P, Q as the external inputs, $\sigma()$ is the sigmoid activation function





FP and stability of the WC model

1. Let $u_E = w_{EE}E - w_{EI}I + P$
$u_I = w_{IE}E - w_{II}I + Q$

2. Fixed point is where: $\left(\dfrac{dE}{dt} = 0, \dfrac{dI}{dt} = 0\right)$

3. So $E^* = -E + \sigma(w_{EE}E^* - w_{EI}I^* + P)$
$I^* = -I + \sigma(w_{IE}E^* - w_{II}I^* + Q)$

4. Stability: linearise around the fixed point

$$\frac{d\mathbf{x}}{dt} \approx \mathcal{J}(E^*, I^*)(\mathbf{x} - \mathbf{x}^*),$$

5. Where

$$\mathcal{J}(E, I) = \begin{pmatrix} \dfrac{-1 + w_{EE}\sigma'(u_E)}{\tau_E} & \dfrac{-w_{EI}\sigma'(u_E)}{\tau_E} \\ \dfrac{w_{IE}\sigma'(u_I)}{\tau_I} & \dfrac{-1 - w_{II}\sigma'(u_I)}{\tau_I} \end{pmatrix}.$$

6. The eigenvalues λ are computed by solving:
$\det(\mathcal{J} - \lambda\mathbb{I}_n) = 0.$

7. The fixed point is ***stable*** if and only if the real part of all eigen values are negative, otherwise it is ***unstable.***

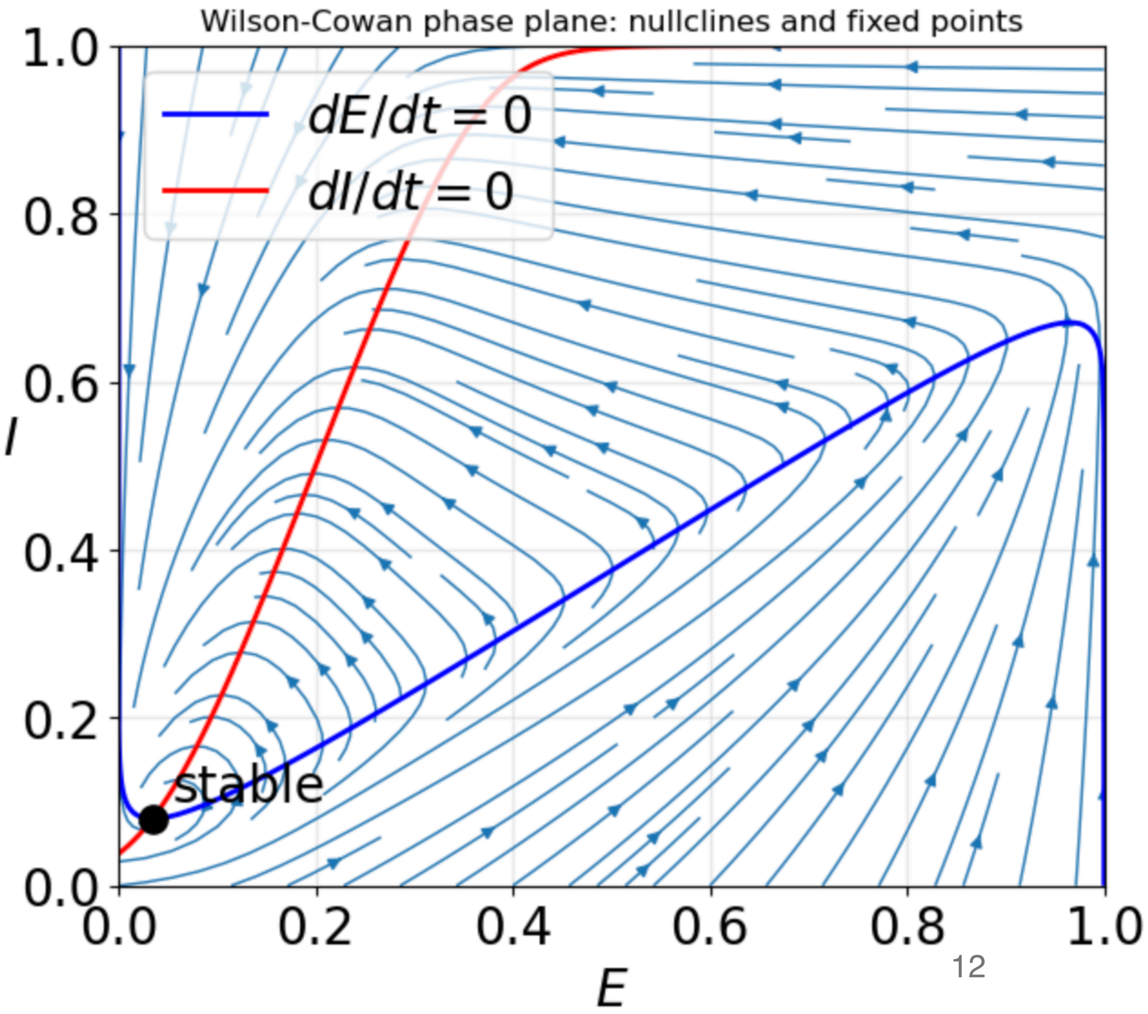
8. See jupyter Notebook

Phase plane analysis

1. Phase plane: plane where we plot the dynamics, for example the $(E, I)-$plane.

2. At each point (E, I) the WC-equations tell us the direction of motion: $\left(\frac{dE}{dt}, \frac{dI}{dt}\right)$. This pair of numbers defines a vector. A ***vector field*** plots these arrows across the whole plane

3. A null cline is a curve where one variable stops changing,
   $E-$null cline: $\frac{dE}{dt} = 0$, $I-$null cline: $\frac{dI}{dt} = 0$



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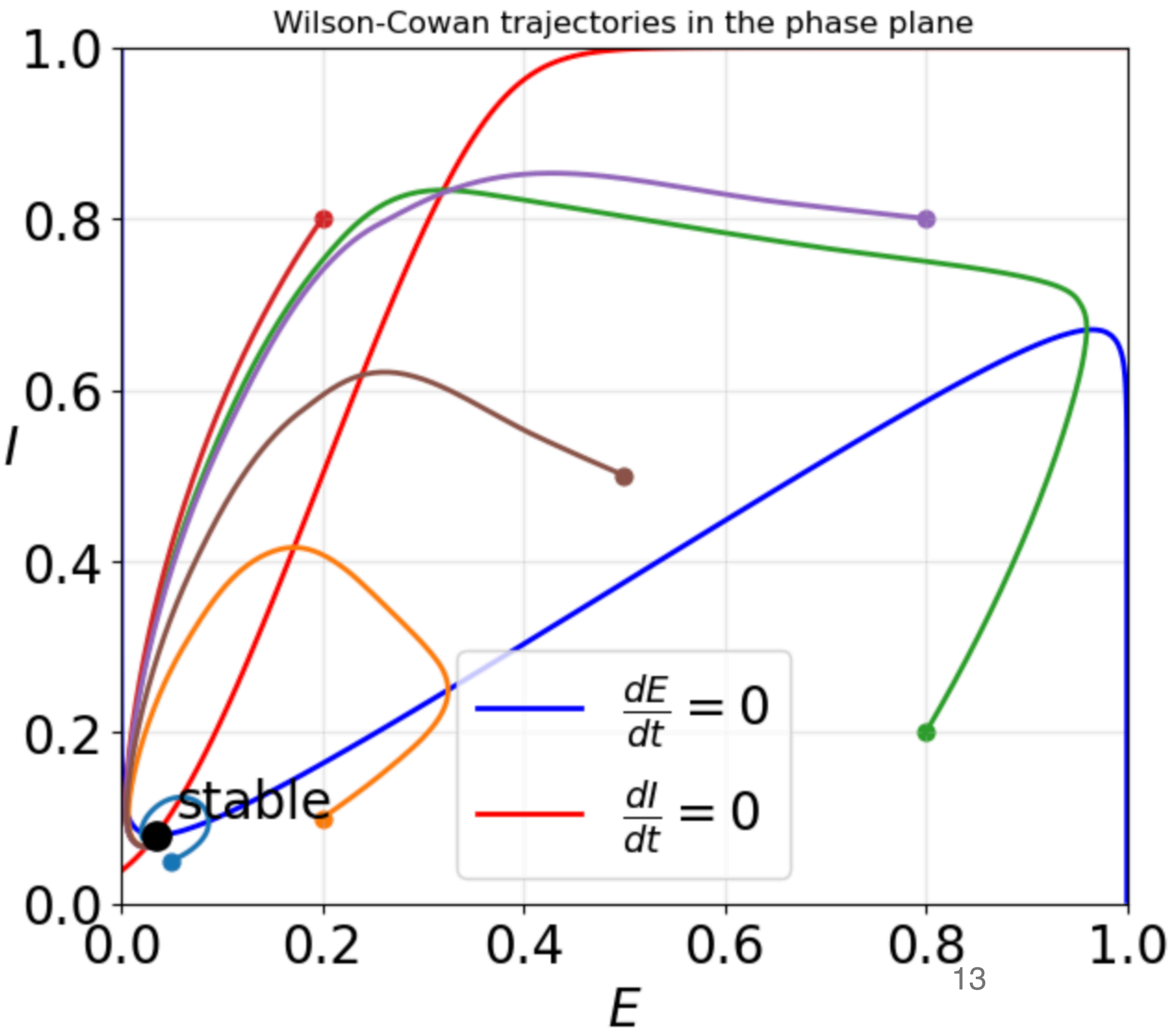


Exercises:

1. Consider the simple model

$$\frac{dx}{dt} = x(1-x),$$

(a) find the fixed points

(b) compute the derivative
at each fixed point

(c) classify each fixed point as stable or unstable

(d) interpret what happens when the initial condition $0 < x(0) < 1$.

1. Consider the model

$$\frac{dE}{dt} = -E - I + 1, \qquad \frac{dI}{dt} = E - 2I\,.$$

(a) find the fixed point (E^*, I^*)

(b) compute the Jacobian matrix

$$\mathscr{J} = \begin{pmatrix} \frac{\partial f}{\partial E} & \frac{\partial f}{\partial I} \\ \frac{\partial g}{\partial E} & \frac{\partial g}{\partial I} \end{pmatrix}.$$

(c) find the eigenvalues of $\mathscr{J}$,

(d) classify the fixed point.

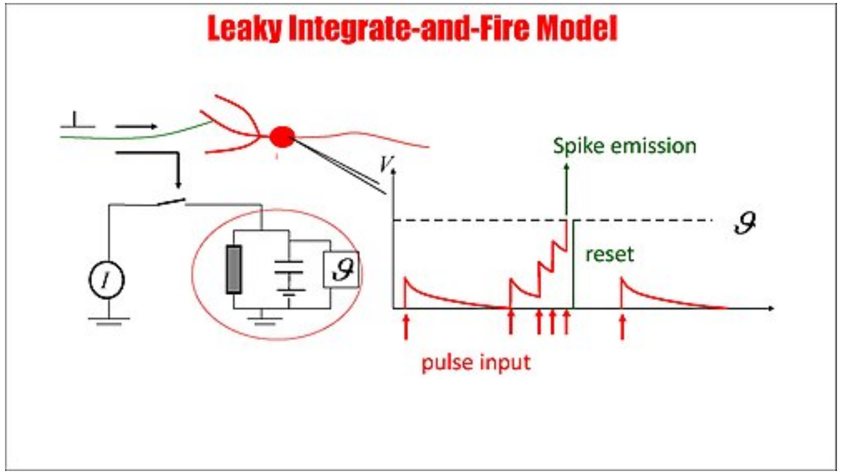
Spiking neural model

1. A popular one is the Leaky Integrate and Fire (LIF) model:

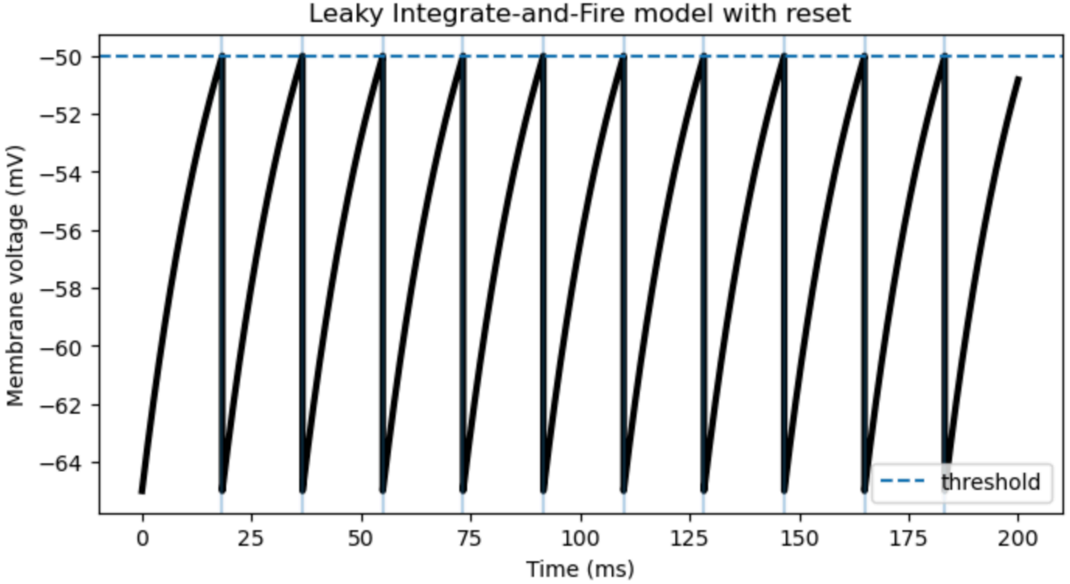
$$\tau_m \frac{dV}{dt} = -(V - V_{\text{rest}}) + RI,$$

$V(t)$: membrane voltage
V_{rest}: resting potential
R: membrane resistance
I : input current
τ_m: membrane time constant

2. Reset: If $V \geq V_{\text{threshold}}$ then $V \leftarrow V_{\text{reset}}$



Source: https://en.wikipedia.org/wiki/Biological_neuron_model



Spiking neural network

1. A spiking neural network describes individual neurons using membrane voltage and spike times

2. For neuron i a LIF model is

$$\tau_m \frac{dV_i}{dt} = -(V_i - V_{\text{rest}}) + RI_i(t),$$
$$i = 1,\ldots,N$$

3. When $V_i(t) \geq V_{\text{th}},\ V_i \leftarrow V_{\text{reset}}\,.$

4. The recurrent input to neuron i: $I_i(t) = I_i^{\text{ext}} + I_i^{\text{syn}}(t)$

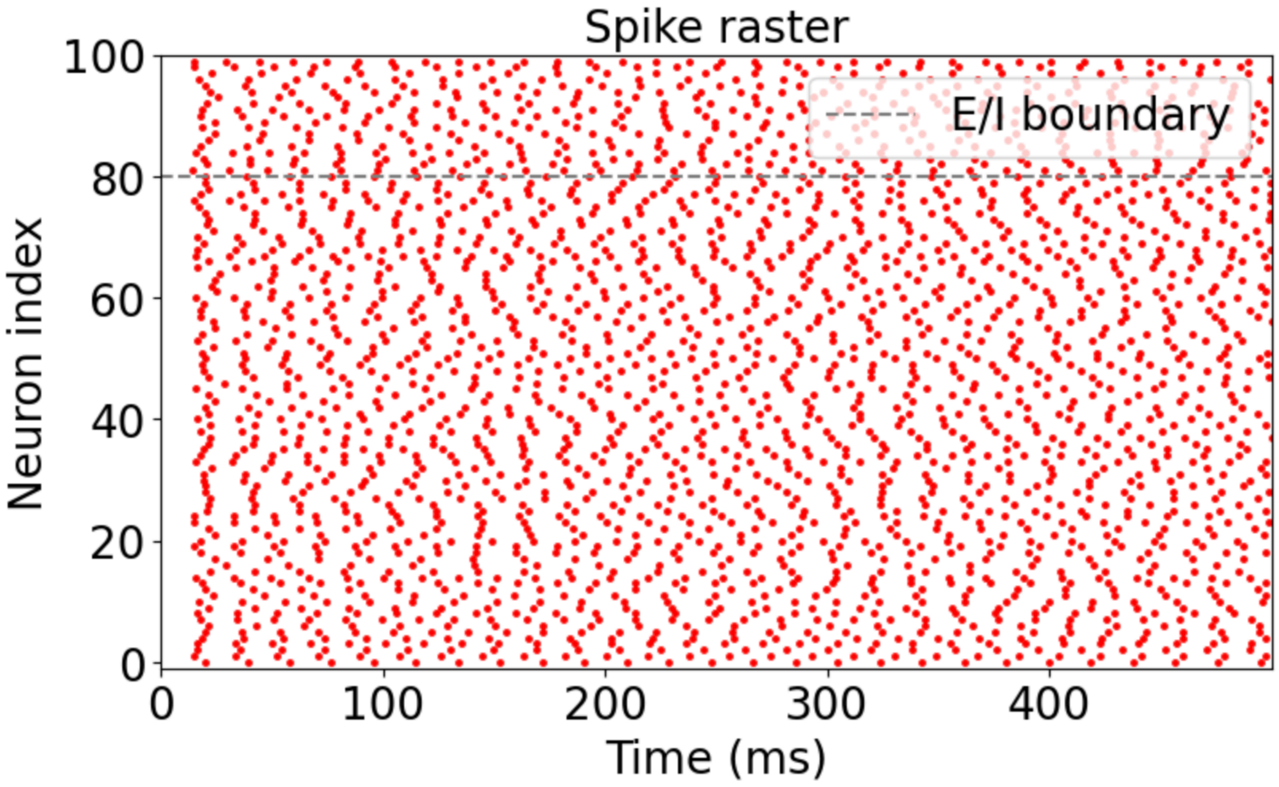
5. Here $I_i^{\text{syn}}(t) = \sum_{j=1}^{N} W_{ij} s_j(t)$, with W_{ij} as the weight from neuron j to i.

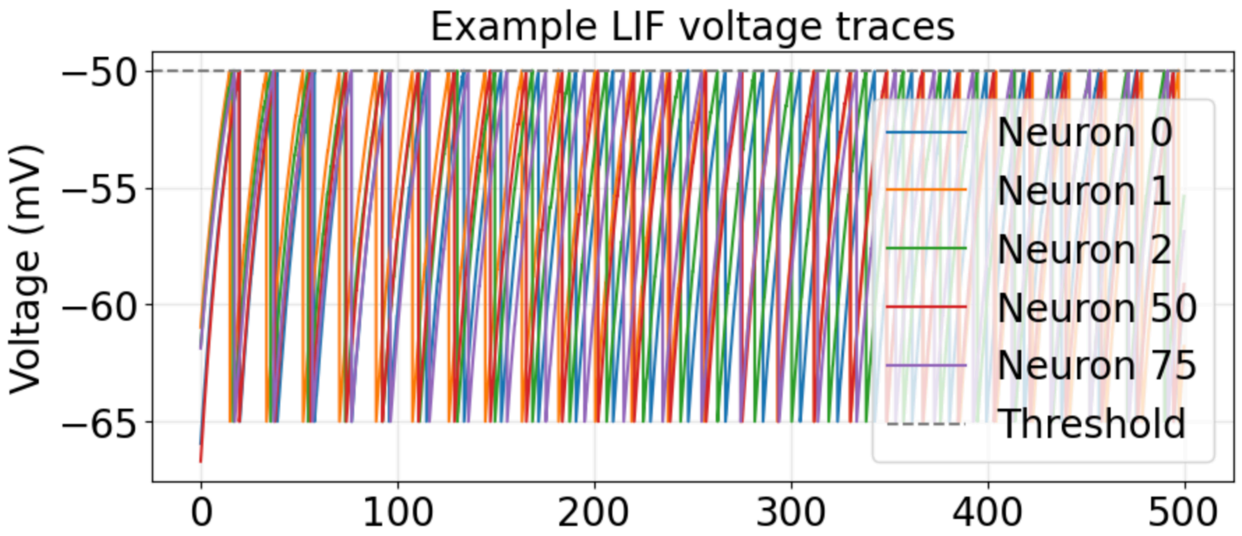
6. Also, $s_j(t) = 1$ if neuron j spiked at time t, else 0.

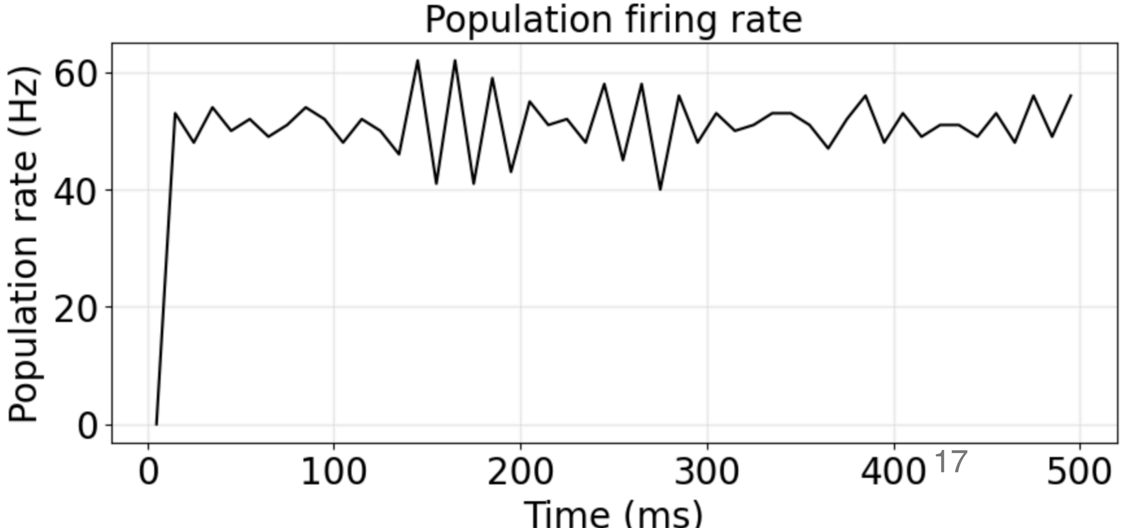
7. We often take 80% excitatory neurons and 20% inhibitory ones. $W_{ij}^E > 0,\ W_{ij}^I < 0$: creates different firing regimes

8. See jupyter Notebook

Spiking neural network


Spike raster


Example LIF voltage traces


Population firing rate

Mean field model

1. If N is large, it becomes hard computationally, and hard to understand every neuron individually.

2. Mean field theory replaces the full network by an average population variable:

$$m(t) = \frac{1}{N}\sum_{i=1}^{N} r_i(t)$$

3. Any interacting neuorns → one average population variable: ***mean field***

For the rate neural network:

$$\tau \frac{dr_i}{dt} = -r_i + \sigma\left(\sum_{j=1}^{N} W_{ij} r_j + I\right),$$

If all neurons are statistically similar,

$$m(t) = \frac{1}{N}\sum_{i=1}^{N} r_i(t).$$

Let the average coupling is J, so that

$$\sum_{j=1}^{N} W_{ij} r_j \approx Jm.$$

Mean field model

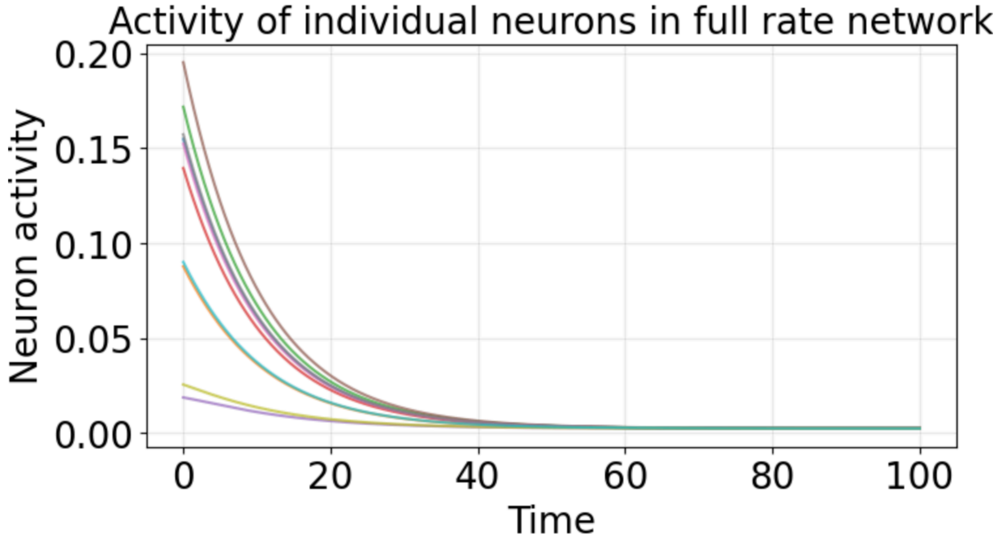
1. Mean field rate neural network:

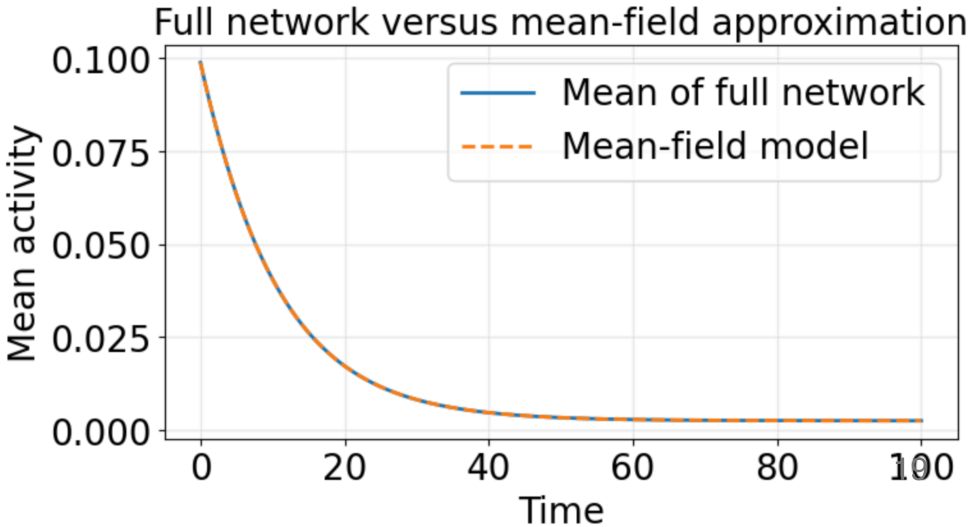
$$\tau \frac{dm}{dt} = -m + \sigma(Jm + I)$$

2. Two population mean field model:

$$\tau_E \frac{dm_E}{dt} = -m_E + \sigma(J_{EE}m_E - J_{EI}m_I + P_E),$$

$$\tau_I \frac{dm_I}{dt} = -m_I + \sigma(J_{IE}m_E - J_{II}m_I + P_I)$$


Activity of individual neurons in full rate network


Full network versus mean-field approximation

Mean field model

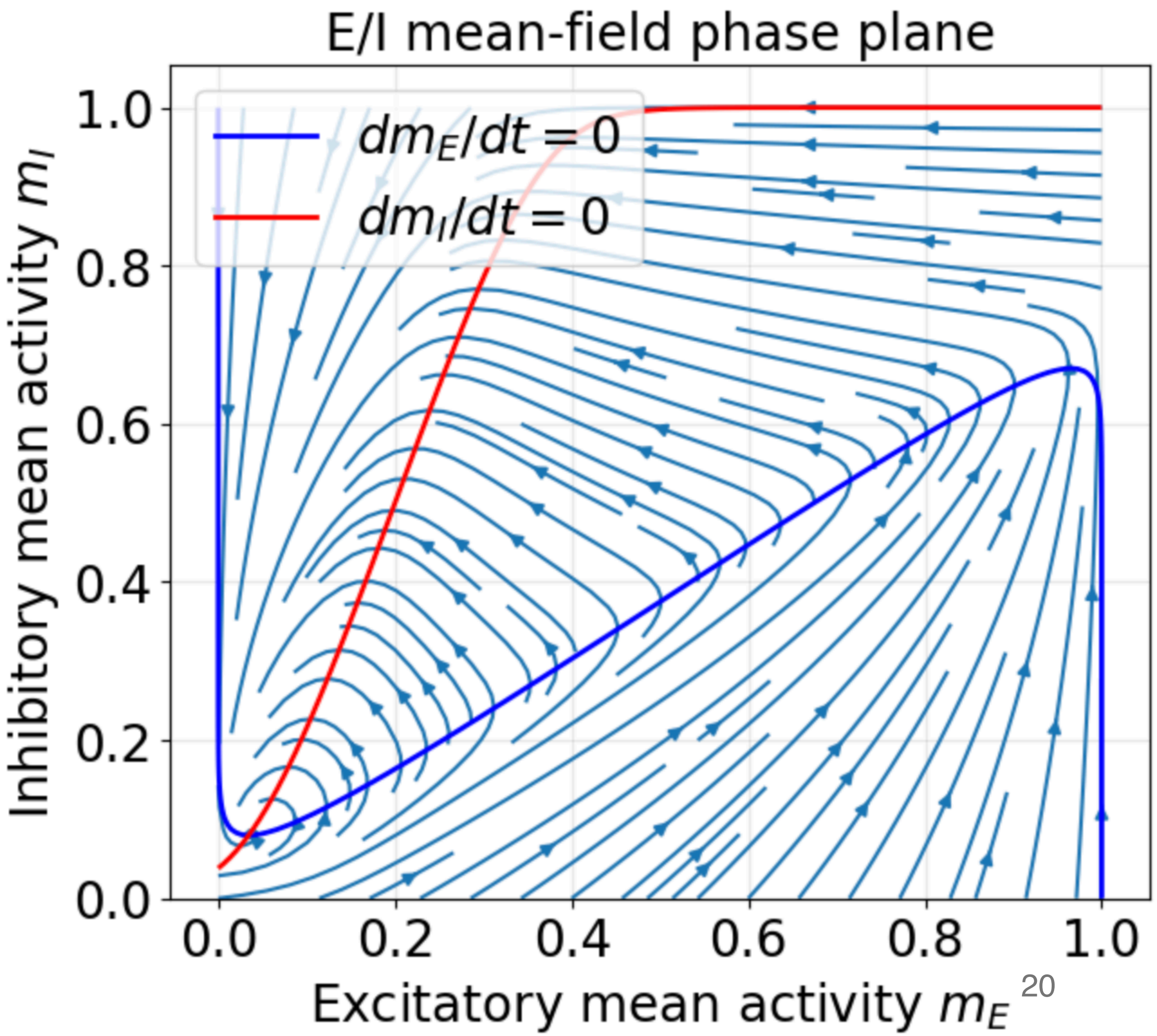
1. Mean field rate neural network:

$$\tau \frac{dm}{dt} = -m + \sigma(Jm + I)$$

2. Two population mean field model:

$$\tau_E \frac{dm_E}{dt} = -m_E + \sigma(J_{EE}m_E - J_{EI}m_I + P_E),$$

$$\tau_I \frac{dm_I}{dt} = -m_I + \sigma(J_{IE}m_E - J_{II}m_I + P_I)$$



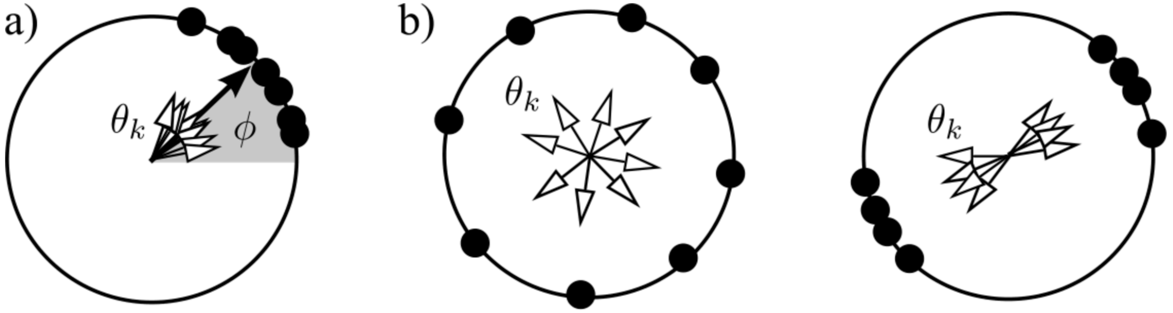
Mean field model

1. Kuramoto's model: for the i-th oscillator in a system of N neurons:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

2. Here θ: phase of the $i-$th oscillator, ω: intrinsic natural frequency, K: coupling strength.

3. "Mean field": The macroscopic order parameter $Re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$. $R(t) \in [0,1]$ measures the phase coherence. $\psi(t)$: average phase of the population.



$0 \ll R < 1$

$R \approx 0$

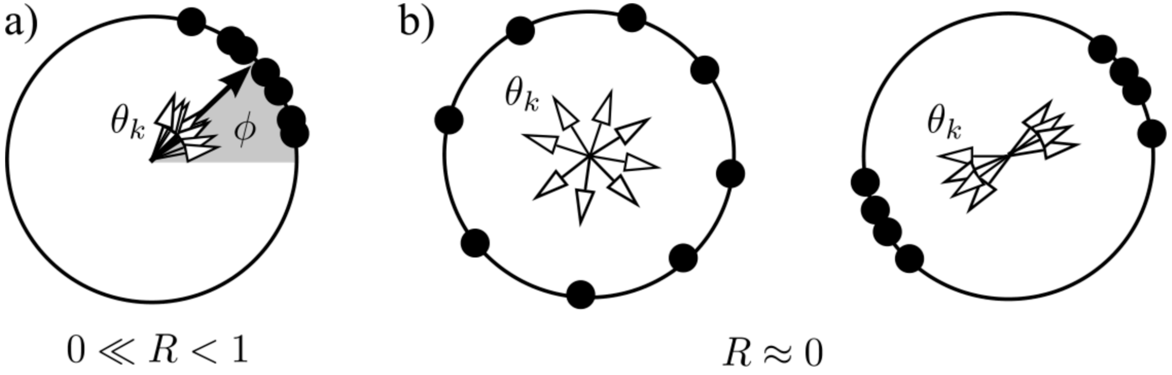
Source: Bick, Christian, et al. "Understanding the dynamics of biological and neural oscillator networks through exact mean-field reductions: a review." *The Journal of Mathematical Neuroscience* 10.1 (2020): 9.

Mean field model

4. By multiplying both sides by $e^{-i\theta}$, and matching the imaginary parts, the $N-$body equation interacting with global field :

$$\dot{\theta} = \omega_i + KR\sin(\psi - \theta_i)$$

5. Responds to ***global mean field*** characterised by R and ψ.



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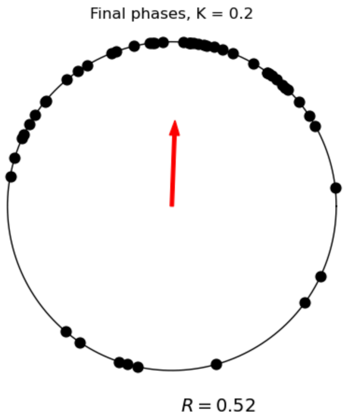

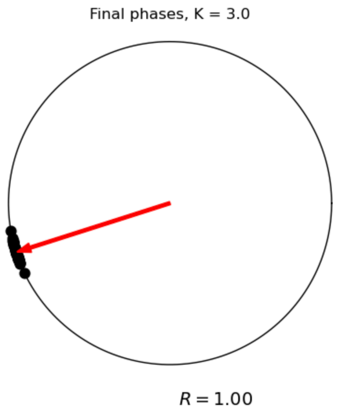

Questions? I’m all ears